Hamiltonian formulation of the Grosse-Wulkenhaar $\phi^4_{\star D}$ model

Mahouton Norbert Hounkonnou, Dine Ousmane Samary and Villevo Adanhounme

University of Abomey-Calavi
International Chair of Mathematical Physics and Applications
ICMPA-UNESCO Chair
072 B.P. 50 Cotonou, Republic of Benin
E-mail: norbert.hounkonnou@cipma.uac.bj[1]

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Abstract

A Hamiltonian formulation for the Grosse-Wulkenhaar $\phi^4_{\star D}$ model is performed. The study is based on $D+1$ dimensional space-time formulation of D dimensional non-local theories. The analysis of constraints shows that the secondary constraints describe the Euler-Lagrange equations of motion. Relevant tensors are computed and analyzed.

Key-words: Noncommutative field theory; Hamiltonian formalism; Grosse-Wulkenhaar model; Energy momentum tensor; Primary and secondary constraints.

[1]With copy to hounkonnou@yahoo.fr

1 Introduction

The search of a unifying theory of gravity and quantum field theory and the obtaining of a better understanding of physics at short distances have led to the development of the noncommutative geometry. Subsequently, non-commutative field theories and quantum gravity have been studied extensively. Such an approach should lead to change the nature of spacetime in a fundamental way. The noncommutativity can be realized through the coordinates which satisfy the commutation relations $[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}(\hat{x})$. $\Theta^{\mu\nu}(\hat{x})$ is unknown, but, for physical reasons, should vanish at large distances where we experience the commutative world and may be determined by experiments. See [1] and [2] and references therein. The algebra $\mathcal{M}$ of functions of such noncommuting coordinates can be represented by the algebra of functions on ordinary spacetime, equipped with a noncommutative $\star-$product. A simple case of this deformation is the D-dimensional Moyal space $\mathbf{R}^D_\Theta$ endowed with a constant Moyal $\star-$bracket of coordinate functions

$$[x^\mu, x^\nu]_\star = i\Theta^{\mu\nu} \tag{1.1}$$

where Θ is a constant $D \times D$ non-degenerate skew-symmetric matrix (which requires D even), usually chosen in the form

$$\Theta = \theta J \text{ with } J = \begin{pmatrix} 0 & I_{\frac{D}{2}} \\ -I_{\frac{D}{2}} & 0 \end{pmatrix}. \tag{1.2}$$

$]0, +\infty[\ni \theta$ is a square length dimensional parameter, ($[\theta] = [L]^2$), D denoting the space-time dimension, $I_{\frac{D}{2}}$ is the $D/2 \times D/2$ identity matrix. The corresponding product of functions is the associative, noncommutative Moyal-Groenewold-Weyl product, simply called hereafter Moyal product or $\star$-product defined by

$$(f \star g)(x) = \mathrm{m}\left\{e^{i\frac{\Theta^{\rho\sigma}}{2}\partial_\rho \otimes \partial_\sigma} f(x) \otimes g(x)\right\} \quad x \in \mathbf{R}^D_\Theta \qquad \forall f, g \in \mathcal{S}(\mathbf{R}^D_\Theta). \tag{1.3}$$

m is the ordinary multiplication of functions and $\mathcal{S}(\mathbf{R}^D_\Theta)$ - the space of suitable Schwartz functions.

The very process of replacing the point-wise multiplication of functions at the same point by a star-product makes the theory non-local. The star-product contains an infinite number of space-time derivatives and this in turn affects the fundamental causal structure on which all local, point-like quantum field theories are built upon. Non-commutative field theories have infinite number of space-time derivatives and then are non-local. The non-local character of the theory also thanks to the property that

$$(f \star g)(x) \quad = \quad \int d^D y d^D z\, K(x; y, z) f(y) g(z) \tag{1.4}$$

is evaluated through a two-point kernel $K(x; y, z) = \delta(x - y) \star \delta(x - z) = \frac{e^{i(x-y)\Theta^{-1}(y-z)}}{(2\pi)^D det\Theta}$. These theories have peculiar properties due to their acausal behavior and lack of unitarity. The lack of unitarity is due to the fact that $\Theta^{0i} \neq 0, i = 1, 2 \cdots$.

This work is devoted to the construction of a Hamiltonian formulation of the Grosse-Wulkenhaar model, one of the very few renormalizable noncommutative theories. The Hamiltonian formulation of classical field theory, crucial in the quantization procedure, remains a task to be solved in the noncommutative field theories (NCFTs) widely developed in recent years [1]-[19] (and references therein). So far, all attempts to solve this problem have been made before the advent of the new class of renormalizable NCFTs built on the Grosse and Wulkenhaar (GW) ϕ^4 scalar field theory. See [3] and [4] (and references therein) for more details. This paper aims at filling this gap, considering the class of renormalizable GW models treated with a method that generalizes previous construction [3]. The expression of the total Hamiltonian of the system is given. From a space-time Galilean transformation and imposing an additional constraint from the application of the Noether's theorem, the Noether currents are computed as a $U_\star(N)$ gauge currents from $U_\star(N)$ gauge transformations with finite translations, i.e. $g_\epsilon(x) = e^{-i\epsilon^\mu \Theta^{-1}_{\mu\nu} x^\nu} \in U_\star(N)$ such that $g_\epsilon(x) \star f(x) \star g^\dagger_\epsilon(x) = f(x+\epsilon)$. The rotation group of $\mathbb{R}^D$ can be considered as a particular concrete case.

2 Hamiltonian formulation of the NCFTs

In this section, we briefly review the Hamiltonian formulation of NCFTs recently developed by Gomis et al [3]-[4]. We then generalize this formulation by introducing a compact support function $w_h(x)$.

2.1 Quick review of Hamiltonian formulation of NCFTs

This subsection, mainly based on [3] and [4] (and references therein), addresses a Hamiltonian formulation of field theories in a noncommutative space-time. This formulation involves two time coordinates t and λ, and the dynamics in this space is described in such a way that the evolution is local with respect to one of the times. The non-local Lagrangian at time t, $L^{non}(t)$, depends not only on variables at time t but also on ones at different times. In other words, it depends on an infinite number of time derivatives of the position $q_i(t)$. The analogue of the tangent bundle for Lagrangians depending on positions and velocities is now infinite dimensional and can be represented as the space of all possible trajectories. The action is given by

$$S[q] = \int dt\, L^{non}(t) = \int dt\, L([q(t+\lambda)]). \tag{2.1}$$

The functional variational principle can be applied to the action (2.1) to produce the Euler-Lagrange (EL) equation of motion as follows:

$$\frac{\delta S[q]}{\delta q(t)} = \int dt'\, \frac{\delta L^{non}(t')}{\delta q(t)} = 0 \tag{2.2}$$

which must be understood as a functional relation to be fulfilled by physical trajectories. The latters are not obtained as evolution of some given initial conditions. Since the equation of motion is of infinite degree in time derivatives, one should give as initial conditions the value of all these derivatives at some initial time. In other words, we should give the whole trajectory (or part of it) as the initial condition. Let $J = \{q(\lambda), \lambda \in \mathbb{R}\}$ be the space of all possible trajectories. Then the EL equation of motion (2.2) is a Lagrangian constraint defining the subspace $J_R \subset J$ of physical trajectories. In $1+1$ dimensional field theory, we introduce new dynamical variables $\mathcal{Q}(t,\lambda)$ such that

$$\mathcal{Q}(t,\lambda) = q(t+\lambda) = T_t q(\lambda) \tag{2.3}$$

where T_t is the time evolution operator for a given initial trajectory $q(\lambda)$. t is the evolution parameter and λ is a continuous parameter indexing the degrees of freedom. If we denote by $\mathcal{P}(t,\lambda)$ the canonical momentum of $\mathcal{Q}(t,\lambda)$, then the Hamiltonian is defined as

$$H(t,[\mathcal{Q},\mathcal{P}]) = \int d\lambda\, \mathcal{P}(t,\lambda)\mathcal{Q}'(t,\lambda) - \widetilde{L}(t,[\mathcal{Q}]) \tag{2.4}$$

where $\mathcal{Q}'(t,\lambda) = \partial_\lambda \mathcal{Q}(t,\lambda)$ and $\widetilde{L}(t,[\mathcal{Q}])$ is a functional defined by

$$\widetilde{L}(t,[\mathcal{Q}]) = \int d\lambda\, \delta(\lambda)\mathcal{L}(t,\lambda). \tag{2.5}$$

The density $\mathcal{L}(t,\lambda)$ is constructed from the original non-local Lagrangian density $L^{non}(t)$ by replacing $q(t)$ by $\mathcal{Q}(t,\lambda)$, the t-derivatives of $q(t)$ by λ-derivatives of $\mathcal{Q}(t,\lambda)$ and $q(t+\rho)$ by $\mathcal{Q}(t,\lambda+\rho)$. In this construction of the Hamiltonian, λ inherits the signature of the original time t and is a time-like coordinate. $\mathcal{L}(t,\lambda)$ is local in t and is non-local in λ. H depends linearly on $\mathcal{P}(t,\lambda)$ but does not depend on $\dot{\mathcal{Q}}(t,\lambda)$.

The first and second Hamilton equations can be written as:

$$\dot{\mathcal{Q}}(t,\lambda) = \mathcal{Q}'(t,\lambda),\ \dot{\mathcal{P}}(t,\lambda) = \mathcal{P}'(t,\lambda) + \frac{\delta\widetilde{L}(t,[\mathcal{Q}])}{\delta\mathcal{Q}(t,\lambda)} \tag{2.6}$$

where $\dot{\mathcal{Q}}(t,\lambda) = \partial_t\mathcal{Q}(t,\lambda)$. Their solutions are related to those of the EL equations of motion of the original non-local Lagrangian L^{non} if we impose a constraint on the momentum

$$\varphi(t,\lambda) \equiv \mathcal{P}(t,\lambda) - \int d\sigma\, \frac{\epsilon(\lambda)-\epsilon(\sigma)}{2}\frac{\delta\mathcal{L}(t,\sigma)}{\delta\mathcal{Q}(t,\lambda)} \approx 0. \tag{2.7}$$

Here $\epsilon(\lambda)$ is the sign distribution. The symbols "$\equiv$" and "$\approx$ " stand for "strong" and " weak" equalities, respectively. Further constraints are generated by requiring the stability of the primary ones. In the first step, we obtain:

$$\dot{\varphi}(t,\lambda) \equiv \varphi'(t,\lambda) + \delta(\lambda)\int d\sigma\, \frac{\delta\mathcal{L}(t,\sigma)}{\delta\mathcal{Q}(t,0)} \approx 0 \tag{2.8}$$

or simply

$$\dot{\varphi}_0(t,\lambda) \equiv \delta(\lambda) \int d\sigma \, \frac{\delta \mathcal{L}(t,\sigma)}{\delta \mathcal{Q}(t,\lambda)} \approx 0 \tag{2.9}$$

which reduces to the EL equation of motion. Repeating this, we get an infinite set of Hamiltonian constraints. So doing, we are able to describe the original non-local Lagrangian system as a 1+1 dimensional local (in one of the times) Hamiltonian system, governed by the Hamiltonian H and a set of constraints. Note that this formalism can be viewed as a generalization of the Ostrogradski construction in the case of infinite order derivative theories.

2.2 A generalization in $1+1$ dimensional field theory

In this subsection, we aim at enlarging the class of Hamiltonians that can be constructed in the framework of the above mentioned formalism. The corresponding system of Hamilton equations and the constraints are deduced.

Consider a parameter $h \in]0,1[, \quad x,\ y \in \mathbb{R}^n$ and define

$$|x-y| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}, \quad \omega_h(x-y) = \frac{\omega(\frac{x-y}{h})}{h^n} \tag{2.10}$$

where

$$\omega(u) = \begin{cases} c \cdot \exp(\frac{1}{|u|^2-1}) & |u| < 1 \\ 0 & |u| \geq 1 \end{cases}, \quad c = \Big[\int_{|u|<1} \exp(\frac{1}{|u|^2-1}) du\Big]^{-1}. \tag{2.11}$$

Then we consider the family of Hamiltonians

$$\mathcal{H}_h(t,[\mathcal{Q}_h,\mathcal{P}_h]) = \int d\lambda \mathcal{P}_h(t,\lambda)\mathcal{Q}'_h(t,\lambda) - L_h(t,[\mathcal{Q}_h]) \tag{2.12}$$

where the quantities $\mathcal{P}_h(t,\lambda)$, $\mathcal{Q}_h(t,\lambda)$, $L_h(t,[\mathcal{Q}_h])$ are defined as follows:

$$\mathcal{P}_h(t,\lambda) = \int_{\mathbb{R}^2} dy \, \omega_h(x-y)\mathcal{P}(y), \;\; \mathcal{Q}_h(t,\lambda) = \int_{\mathbb{R}^2} dy \, \omega_h(x-y)\mathcal{Q}(y) \tag{2.13}$$

$$L_h(t,[\mathcal{Q}_h]) = \int_{\mathbb{R}^2} dy \, \omega_h(x-y) L(t',[Q]) \tag{2.14}$$

$x = (t,\lambda)$ and $y = (t',\lambda')$, $\mathcal{Q}'_h(t,\lambda) = \partial_\lambda \mathcal{Q}_h(t,\lambda)$ and $\int_{\mathbb{R}^n} dy \, \omega_h(x-y) = 1$.

Lemma: *Let $L_h, \quad h \in]0,1[$, define a class of differentiable functionals with compact support, i.e. for $|x| < M, \quad L_h(x) \neq 0$, and $L_h(x) = 0$ otherwise, where M is a positive number:*

$$L_h(x) = \int_{\mathbb{R}^n} dy \omega_h(x-y) L(y).$$

If the functional L is summable on $x \in \mathcal{D} \subset \mathbb{R}^n$, then

$$\int_{\mathcal{D}} |L_h(x) - L(x)| dx \to 0 \textit{ for } h \to 0, \tag{2.15}$$

i.e. L_h converges in average to L for $h \to 0$ where $\mathcal{D}$ is a bounded measurable set.

Proof: For $x \in \mathcal{D}$, we have

$$L_h(x) - L(x) = \int_{\mathbb{R}^n} \omega_h(x-y)[L(y) - L(x)]\mathrm{d}y \tag{2.16}$$

where

$$L(x) = \int_{\mathbb{R}^n} \omega_h(x-y)L(x)\mathrm{d}y = \frac{1}{h^n} \int_{|x-y|\leq h} \omega\Big(\frac{x-y}{h}\Big) L(x) dy.$$

Then

$$|L_h(x) - L(x)| \leq \tfrac{c_1}{h^n} \textstyle\int_{|x-y|\leq h} |L(y) - L(x)| dy, \tag{2.17}$$

$$\text{with} \qquad c_1 = \max_{|x-y|\leq h} \omega\Big(\tfrac{x-y}{h}\Big). \tag{2.18}$$

Using Fubini's theorem we can get

$$\begin{aligned} \int_{\mathcal{D}} |L_h(x) - L(x)| dx &\leq \frac{c_1}{h^n} \int_{\mathcal{D}} \int_{|x-y|\leq h} |L(y) - L(x)| dy dx \\ &\leq \frac{c_1}{h^n} \int_{|y'|\leq h} \mathrm{d}y' \int_{\mathcal{D}} |L(x+y') - L(x)| \mathrm{d}x. \end{aligned} \tag{2.19}$$

By the mean-continuity property, i.e. for all small $\epsilon > 0$ there exists a small $\delta > 0$ such that

$$\int_{\mathcal{D}} |L(x+y') - L(x)| \mathrm{d}x \leq \epsilon, \text{ for } |y'| \leq \delta, \tag{2.20}$$

we obtain

$$\int_{\mathcal{D}} |L_h(x) - L(x)| \mathrm{d}x \leq \frac{c_1 \epsilon}{h^n} \int_{|y'|\leq h} dy' \Rightarrow \int_{\mathcal{D}} |L_h(x) - L(x)| \mathrm{d}x \leq c_1 \epsilon \int_{|z|\leq 1} \mathrm{d}z$$
$$\Rightarrow \int_{\mathcal{D}} |L_h(x) - L(x)| \mathrm{d}x \leq c_1 \cdot c_2 \epsilon$$

where c_2 is the volume of the unit sphere in $\mathbb{R}^n$. $\square$

Consider now the Lagrangian density $\mathcal{L}_h(t, \lambda)$ defined by

$$\mathcal{L}_h(t, [\mathcal{Q}_h]) = \int_{\mathbb{R}^2} dy\, \omega(x-y) \mathcal{L}_h(y). \tag{2.21}$$

Following [3], the density $\mathcal{L}_h(t, \lambda)$ is constructed from $\mathcal{L}_h^{non}(t) = L([q_h(t+\lambda)])$ by replacing $q_h(t)$ by $\mathcal{Q}_h(t, \lambda)$, the t-derivatives of $q_h(t)$ by λ-derivatives of $\mathcal{Q}_h(t, \lambda)$ and $q_h(t+\rho)$ by $\mathcal{Q}_h(t, \lambda+\rho)$. $\mathcal{L}_h(t, \lambda)$ is local in t and nonlocal in λ. Defining then a time evolution operator T_t for a given initial trajectory $q(t)$ as follows

$$T_t : q(\lambda) \mapsto q(t+\lambda), \tag{2.22}$$

we introduce a family of new dynamical variables $\mathcal{Q}_h(t,\lambda)$, for $0<h<1$ as:

$$\mathcal{Q}_h(t,\lambda)=q_h(t+\lambda)=:T_t\Big(q_h(\lambda)\Big). \tag{2.23}$$

t is the "evolution" parameter and λ is a continuous parameter indexing the degrees of freedom. In differential form, condition (2.23) reads:

$$\frac{\partial\mathcal{Q}_h}{\partial t}(t,\lambda)=\frac{\partial\mathcal{Q}_h}{\partial\lambda}(t,\lambda). \tag{2.24}$$

The fundamental Poisson bracket turns to be:

$$\{\mathcal{Q}_h(t,\lambda),\mathcal{P}_h(t,\lambda')\}=\omega_h(\lambda-\lambda'). \tag{2.25}$$

The relation (2.24) defines a family of first Hamilton equations for (2.12). The corresponding family of second Hamilton equations can be written as follows:

$$\dot{\mathcal{P}}_h(t,\alpha) \;=\; \mathcal{P}'_h(t,\alpha)+\frac{\partial L_h(t,[\mathcal{Q}_h])}{\partial\mathcal{Q}_h(t,\alpha)} \tag{2.26}$$

where $\mathcal{P}_h(t,\lambda)\omega_h(\lambda-\alpha)\Big|_{\lambda=-M}^{M}=0$ ($\mathcal{P}_h$ with compact support). Now integrating the second Hamilton equations yields

$$\Gamma_h(t,\lambda,[\mathcal{Q}_h,\mathcal{P}_h])\equiv\mathcal{P}_h(t,\lambda)-\int d\sigma\frac{\delta\mathcal{L}_h(t,\sigma)}{\delta\mathcal{Q}_h(t,\lambda)}\cdot\frac{\epsilon(\lambda)-\epsilon(\sigma)}{2}\approx 0. \tag{2.27}$$

The stability of primary constraints implies the secondary constraints given by

$$\Xi_h\equiv\int d\lambda\,\frac{\delta\mathcal{L}_h(t,\lambda)}{\delta\mathcal{Q}_h(t,0)}\approx 0. \tag{2.28}$$

3 Hamiltonian formulation of the Grosse-Wulkenhaar model

In this section, we first recall the GW model, derive the equation of motion and its solution in matrix base and compute the Noether currents. Then we apply the generalized formulation of NC Hamiltonian construction developed in the previous section to the GW model and investigate the corresponding NC currents.

3.1 GW model

Let us briefly recall the GW model and the Euler Lagrange equations of motion with its solution in a matrix base formalism.

The renormalizable GW model is described by the Lagrangian [6]

$$\begin{aligned}\mathcal{L}^{non}(t) \;&=\; \mathcal{L}_\star[\phi,\partial_\mu\phi]=\frac{1}{2}\partial_\mu\phi(x)\star\partial^\mu\phi(x)+\frac{\Omega^2}{2}\Big(\tilde{x}_\mu\phi(x)\Big)\star\Big(\tilde{x}^\mu\phi(x)\Big)\\ &+\;\frac{m^2}{2}\phi(x)\star\phi(x)+\frac{\lambda}{4!}\phi(x)^4_\star,\end{aligned} \tag{3.1}$$

where $\tilde{x}_\mu = 2(\Theta^{-1})_{\mu\nu}x^\nu$ and $\phi^n_\star = \underbrace{\phi\star\phi\cdots\star\phi}_{\text{n times}}$. Θ breaks into diagonal blocks $\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$. The harmonic term Ω ensures ultraviolet (UV)/infrared (IR) freedom for the action implying its renormalizability, and such that the Lagrangian action becomes covariant under Langmann-Szabo duality [10], i.e. covariant under the symmetry: $\tilde{x}_\mu \longleftrightarrow p_\mu \equiv \partial_\mu$ giving

$$\mathcal{S}[\phi, m, \lambda, \Omega] \to \Omega^2 \mathcal{S}[\phi, \frac{m}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega}]. \tag{3.2}$$

The Lagrangian density depending explicitly on x^μ, through the field ϕ interaction with a harmonic external source, does not describe a closed system. Furthermore, it is not invariant under space-time translation. Besides, at the parameter limit $\theta \to 0$, the model does not converge to the ordinary ϕ^4 scalar field theory due to the presence of the inverse matrix (Θ^{-1}), then causing a singularity. The $\star$-Grosse-Wulkenhaar ϕ^4_D theory is renormalizable at all orders in λ. This result has been now proved by various methods (see [22] and references therein). For more details on the properties of this model, see [6], [14] and [16] (and references therein). The $\star$-product of two real fields is not necessarily real, and the non-locality comes from the infinite derivatives in (1.3).

The peculiar EL equations of motion can be readily derived for the field ϕ by direct application of the variational principle. We get

$$\frac{\delta\mathcal{S}}{\delta\phi} = \int d^Dx \Big(-\partial_\rho\partial^\rho\phi + m^2\phi + \frac{\lambda}{3!}\phi^3_\star + \Omega^2\, \tilde{x}\star\tilde{x}\star\phi\Big) = 0$$

which gives, using the identity $\int d^Dx\,(f\star g) = \frac{1}{2}\int d^Dx\,(f\star g + g\star f)$,

$$\frac{\delta\mathcal{S}}{\delta\phi} = 0 \Longleftrightarrow -\partial_\rho\partial^\rho\phi + m^2\phi + \frac{\lambda}{3!}\phi^3_\star + \frac{\Omega^2}{4}\Big(2\tilde{x}\star\phi\star\tilde{x} + \{\phi, \tilde{x}\star\tilde{x}\}_\star\Big) = 0, \tag{3.3}$$

where $\{.,.\}_\star$ defines the star anticommutator.

Remark that from the equation

$$\frac{\delta\mathcal{S}}{\delta\tilde{x}_\nu} = 0 \Leftrightarrow \Omega^2(2\phi\star\tilde{x}^\nu\star\phi + \tilde{x}^\nu\star\phi^2_\star + \phi^2_\star\star\tilde{x}^\nu) + \Theta^{\mu\nu}\partial_\mu\phi\star\frac{\delta\mathcal{S}}{\delta\phi} = 0 \tag{3.4}$$

we get the additional constraint

$$\pi(\phi, \tilde{x}) = \Omega^2(2\phi\star\tilde{x}^\nu\star\phi + \tilde{x}^\nu\star\phi^2_\star + \phi^2_\star\star\tilde{x}^\nu) \approx 0. \tag{3.5}$$

de Goursac *et al* [23] solved the equation of motion (3.3), representing the elements on the $D-$dimensional Moyal algebra $\mathcal{M}$ with the help of a matrix base [19] whose elements

$b_{kl}^{(D)}(x)$ are eigenfunctions of the harmonic oscillator Hamiltonian

$$H = \sum_{l=1}^{\frac{D}{2}} \frac{1}{2}\Big(x_{2l-1}^2 + x_{2l}^2\Big),$$

with $b_{00}^{(D)} = 2^{D/2}e^{-2H/\theta}$ such that $b_{00}^{(D)} \star b_{00}^{(D)} = b_{00}^{(D)}$. Then defining the operators

$$a_l = \frac{x_{2l-1} + ix_{2l}}{\sqrt{2}}, \text{ and } \bar{a}_l = \frac{x_{2l-1} - ix_{2l}}{\sqrt{2}}$$

together with

$$b_{kl}^{(D)} = \frac{\bar{a}_\star^k \star b_{00}^{(D)} \star a_\star^l}{\sqrt{k!l!\theta^{|k|+|l|}}} \tag{3.6}$$

where $a = \sum_{i=1}^{D/2} a_i$, and $\bar{a} = \sum_{i=1}^{D/2} \bar{a}_i$, one has the following left and right actions:

$$a \star b_{kl}^{(D)} = \sqrt{|k|\theta} b_{k-1,l}^{(D)}, \quad b_{kl}^{(D)} \star a = \sqrt{\theta(|l+1|)} b_{k,l+1}^{(D)}, \tag{3.7}$$

$$\overline{a} \star b_{kl}^{(D)} = \sqrt{\theta(|k+1|)} b_{k+1,l}^{(D)}, \quad b_{kl}^{(D)} \star \overline{a} = \sqrt{|l|\theta} b_{k,l-1}^{(D)} \tag{3.8}$$

and

$$H \star b_{kl}^{(D)} = \theta(|k| + \frac{1}{2}) b_{kl}^{(D)}, \quad b_{kl}^{(D)} \star H = \theta(|l| + \frac{1}{2}) b_{kl}^{(D)}, \tag{3.9}$$

where $k, l \in \mathbb{N}^{D/2}$ and $|k| = \sum_{i=1}^{D/2} k_i$. For $D = 2$, $b_{kl}^{(2)} = f_{kl}$ which can be expanded in polar coordinates, $(x_1 = rcos(\varphi), \quad x_2 = rsin(\varphi))$, to give

$$f_{kl} = 2(-1)^k \sqrt{\frac{k!}{l!}} e^{i(l-k)\varphi} \Big(\frac{2r^2}{\theta}\Big)^{\frac{l-k}{2}} L_k^{l-k}\Big(\frac{2r^2}{\theta}\Big) e^{-\frac{r^2}{\theta}} \tag{3.10}$$

where the $L_n^k(x)$ are the associated Laguerre polynomials. The generalization to higher dimensions is straightforward. In particular, for $D = 4$, one gets $k = (k_1, k_2)$, $l = (l_1, l_2)$ and

$$b_{kl}^{(4)}(x) = f_{k_1,l_1}(x_1, x_2) f_{k_2,l_2}(x_3, x_4).$$

More generally, the following properties are satisfied:

$$(b_{kl}^{(D)} \star b_{k'l'}^{(D)})(x) = \delta_{lk'} b_{kl'}^{(D)}(x), \tag{3.11}$$

$$\int d^D x \; b_{kl}^{(D)}(x) = (2\pi\theta)^{D/2} \delta_{kl}, \tag{3.12}$$

$$(b_{kl}^{(D)})^\dagger = b_{lk}^{(D)}. \tag{3.13}$$

The existence of an isomorphism between the unital involutive Moyal algebra and a subalgebra of the unital involutive algebra of complex infinite-dimensional matrices allows to define, for all $g \in \mathcal{M}$, a unique matrix (g_{kl}) given by

$$g_{kl} = \frac{1}{(2\pi\theta)^{D/2}} \int d^D x \; g(x) b_{kl}^{(D)}$$

satisfying

$$\forall x \in \mathbb{R}^D, \quad g(x) = \sum_{k,l \in \mathbb{N}^{D/2}} g_{kl}\, b_{kl}^{(D)}(x).$$

Setting $\phi(x) = \tau b_{kl}^{(D)}(x)$, where τ is a complex constant, considering

$$\tilde{x} \star \tilde{x} \star b_{kl}^{(D)} = -\frac{8}{\theta^2} H \star b_{kl}^{(D)} = -\frac{8}{\theta}(|k| + \frac{1}{2}) b_{kl}^{(D)}, \tag{3.14}$$

and

$$\begin{aligned} \tilde{x} \star b_{kl}^{(D)} \star \tilde{x} &= -\frac{4}{\theta^2}(a \star b_{kl}^{(D)} \star \bar{a} + \bar{a} \star b_{kl}^{(D)} \star a) & (3.15)\\ &= -\frac{4}{\theta}\Big(\sqrt{|k||l|} b_{k-1,l-1}^{(D)} + \sqrt{|k+1||l+1|} b_{k+1,l+1}^{(D)}\Big), & (3.16) \end{aligned}$$

and taking into account the relations

$$[\tilde{x}_\mu, \phi]_\star = 2i\partial_\mu \phi \Rightarrow \partial_\mu \partial^\mu \phi = -\frac{1}{4}[\tilde{x}_\mu, [\tilde{x}_\mu, \phi]_\star]_\star,$$

the equation of motion (3.3) can be rewritten in the form

$$\frac{1}{2}(\Omega^2 - 1)\tilde{x} \star \phi \star \tilde{x} + \frac{1}{4}(\Omega^2 + 1)\Big(\tilde{x} \star \tilde{x} \star \phi + \phi \star \tilde{x} \star \tilde{x}\Big) + m^2 \phi + \frac{\lambda}{3!}\phi_\star^3 = 0 \tag{3.17}$$

or equivalently

$$\begin{aligned} &-\frac{2}{\theta}(\Omega^2 - 1)\Big(\sqrt{|k||l|} b_{k-1,l-1}^{(D)} + \sqrt{|k+1||l+1|} b_{k+1,l+1}^{(D)}\Big) \\ &-\frac{2}{\theta}(\Omega^2 + 1)\Big(|k| + |l| + |1|\Big) b_{kl}^{(D)} + \Big(m^2 + \frac{\lambda}{3!}\tau^2\Big) b_{kl}^{(D)} = 0. \end{aligned} \tag{3.18}$$

If $\Omega = 1$, then (3.18) is reduced to

$$\Big[-\frac{4}{\theta}\Big(|k| + |l| + |1|\Big) + \Big(m^2 + \frac{\lambda}{3!}\tau^2\Big)\Big] b_{kl}^{(D)} = 0. \tag{3.19}$$

The vectors k and l can be chosen such that $\frac{4}{\theta}\Big(|k| + |l| + |1|\Big) \geq m^2$. In this case, the suitable solution is $\tau = \Big(\frac{3!}{\lambda}\Big)^{1/2}\Big[\frac{4}{\theta}\Big(|k| + |l| + |1|\Big) - m^2\Big]^{1/2}$ and finally

$$\phi(x) = \Big(\frac{3!}{\lambda}\Big)^{1/2}\Big[\frac{4}{\theta}\Big(|k| + |l| + |1|\Big) - m^2\Big]^{1/2} b_{kl}^{(D)}(x). \tag{3.20}$$

More details can be found in [23].

3.2 Noether currents

Let us now consider a Lie group of continuous transformations

$$x^\mu \longmapsto x'^\mu = x'^\mu(x)$$

$$\phi(x) \longmapsto \phi'(x') = \phi'(\phi(x), x)$$

inducing a set of infinitesimal transformations

$$\delta x^\mu(\omega,\epsilon) = \omega^\mu_\nu x^\nu + \epsilon^\mu, \;\; \omega_{\mu\nu} = -\omega_{\nu\mu} \tag{3.21}$$

$$\delta_\pm\phi(\omega,\epsilon,\mathcal{X},\xi) = \mathcal{X}\star\phi + \phi\star\xi + (\delta x^\mu \star \partial_\mu\phi)_\pm, \tag{3.22}$$

where $\omega_{\mu\nu}$ is an antisymmetric constant matrix, ϵ^μ a constant vector. $\mathcal{X} = \mathcal{X}(x)$ and $\xi = \xi(x)$ are two families of functions, spanning the Lie algebra of the Lie group of fixed dimension r; $\Big((f\star g)_+ = f\star g, \; (f\star g)_- = g\star f\Big)$.The transformations (3.21) and (3.22) actually meet known infinitesimal transformations in NCFTs. Indeed, generators of deformed Poincaré or Galilean transformations are recovered for $\mathcal{X} = 0 = \xi$.

$$\delta_\pm(.) = (\delta x^\mu \star \partial_\mu(.))_\pm, \;\; \delta_\pm\phi = (\omega^\mu_\nu x^\nu \star \partial_\mu)_\pm + \epsilon^\mu\partial_\mu\phi. \tag{3.23}$$

In this situation, a $\star-$deformed Poincaré or Galilean algebra can be defined by the generators $p_\mu(.) = \partial_\mu(.)$ and $m_\pm(\omega)(.) = (\omega^\mu_\nu x^\nu \star \partial_\mu(.))_\pm$ satisfying

$$[p_\mu, m_\pm(\omega)](.) = \omega^\nu_\mu p_\nu, \tag{3.24}$$

$$[m_\pm(\omega), m_\pm(\omega')](.) = (\omega\times\omega')^\nu_\rho (x^\rho \star \partial_\nu(.))_\pm = \delta_\pm(\omega\times\omega')(.) \tag{3.25}$$

$$(\omega\times\omega')^\nu_\rho := -(\omega^\nu_\mu \omega'^\mu_\rho - \omega'^\nu_\mu \omega^\mu_\rho). \tag{3.26}$$

Besides, pure translation symmetry is obviously obtained by setting $\omega = 0$ in (3.23). A finite dimensional group of transformations can be obtained by simple exponentiation of these infinitesimal generators. From the infinitesimal transformation (3.23), we can now define the generalized global Ward identity operator (WIop) as follows [11], [14]:

$$\mathcal{W} \;\; = \;\; \frac{1}{2}\int d^Dx\,(\delta\phi\star\frac{\delta(.)}{\delta\phi} + \frac{\delta(.)}{\delta\phi}\star\delta\phi + \delta\tilde{x}_\rho\star\frac{\delta(.)}{\delta\tilde{x}_\rho} + \frac{\delta(.)}{\delta\tilde{x}_\rho}\star\delta\tilde{x}_\rho) \tag{3.27}$$

such that its action on the Lagrangian action gives, (after lengthy but straightforward computations),

$$\mathcal{WS} \;\; = \;\; \int d^Dx\Big[-\epsilon^\mu\partial^\rho\mathcal{T}_{\rho\mu} - \frac{\omega^{\mu\nu}}{2}\partial^\rho\Big(x_\nu\star\mathcal{T}_{\rho\mu} - x_\mu\star\mathcal{T}_{\rho\nu}\Big) + \mathcal{B}(\omega)\Big] \tag{3.28}$$

where the Galilean invariance breaking term $\mathcal{B}(\omega)$ is given by

$$\mathcal{B}(\omega) \;\; = \;\; -\omega^{\mu\nu}x_\nu\star\Big(\frac{\lambda}{4!}[[\partial_\mu\phi,\phi]_\star,\phi\star\phi]_\star + \frac{\Omega^2}{8}[[\partial_\mu\phi,\{\tilde{x}_\nu,\phi\}_\star]_\star,\tilde{x}^\nu]_\star\Big), \tag{3.29}$$

while the canonical energy momentum tensor $\mathcal{T}_{\rho\mu}$ and the broken angular momentum tensor $\mathcal{M}_{\nu\rho\mu}$ are expressed by the relations

$$\mathcal{T}_{\rho\mu} = \frac{1}{2}\{\partial_\rho\phi,\partial_\mu\phi\}_\star - g_{\rho\mu}\mathcal{L}_\star, \;\; \mathcal{M}_{\nu\rho\mu} = x_\nu\star\mathcal{T}_{\rho\mu} - x_\mu\star\mathcal{T}_{\rho\nu}, \tag{3.30}$$

respectively. $g_{\rho\mu}$ stands for the Euclidean metric. In the particular case where $\omega = 0$ (pure translation), the action becomes invariant and $\mathcal{WS} = 0$.

3.3 Hamiltonian formulation of the GW model in $D+1$ dimensions

We now consider the transformation of the D canonical field variables into the $D+1$ ones

$$x^\mu = (t, x^i) \longmapsto X^\mu = (t, x^0, x^i) = (t, \bar{x}^i),\ \ \phi(x) \longmapsto \mathcal{Q}(t, \bar{x})$$

and $\tilde{x} \longmapsto \tilde{X} = (t, \tilde{\bar{x}}) = (t, 2(\Theta^{-1})\bar{x})$. In this case, the Lagrangian density takes the form

$$\begin{aligned}\mathcal{L}^{non}(t,\bar{x}) &= \frac{1}{2}\partial_\mu \mathcal{Q}(t,\bar{x}) \star \partial^\mu \mathcal{Q}(t,\bar{x}) + \frac{\Omega^2}{2}\Big(\tilde{\bar{x}}_\mu \mathcal{Q}(t,\bar{x})\Big) \star \Big(\tilde{\bar{x}}^\mu \mathcal{Q}(t,\bar{x})\Big) \\ &+ \frac{m^2}{2}\mathcal{Q}(t,\bar{x}) \star \mathcal{Q}(t,\bar{x}) + \frac{\lambda}{4!}\mathcal{Q}(t,\bar{x}) \star \mathcal{Q}(t,\bar{x}) \star \mathcal{Q}(t,\bar{x}) \star \mathcal{Q}(t,\bar{x}).\end{aligned} \tag{3.31}$$

Substituting $\phi(x)$ by $\mathcal{Q}_h(t, x^0, x^i)$, we get the family of Lagrangian densities

$$\begin{aligned}\mathcal{L}_h^{non}(t,\bar{x}) &= \frac{1}{2}\partial_\mu \mathcal{Q}_h(t,\bar{x}) \star \partial^\mu \mathcal{Q}_h(t,\bar{x}) + \frac{m^2}{2}\mathcal{Q}_h(t,\bar{x}) \star \mathcal{Q}_h(t,\bar{x}) \\ &+ \frac{\Omega^2}{2}\Big(\tilde{\bar{x}}_\mu \mathcal{Q}_h(t,\bar{x})\Big) \star \Big(\tilde{\bar{x}}^\mu \mathcal{Q}_h(t,\bar{x})\Big) \\ &+ \frac{\lambda}{4!}\mathcal{Q}_h(t,\bar{x}) \star \mathcal{Q}_h(t,\bar{x}) \star \mathcal{Q}_h(t,\bar{x}) \star \mathcal{Q}_h(t,\bar{x}).\end{aligned} \tag{3.32}$$

In view of computing the constraints, let us define the symmetric Kernel K of four star products by

$$K(x-x_1, x-x_2, x-x_3, x-x_4) = e^{-ix\wedge\sum_{i=1}^{4}(-1)^{i+1}x_i} e^{-i\varphi_4} \tag{3.33}$$

where $\varphi_4 = \sum_{i<j=1}^{4}(-1)^{i+j+1}x_i \wedge x_j$, $x \wedge y = 2x\Theta^{-1}y$. In expanded form, we get

$$\begin{aligned}&K(x-x_1, x-x_2, x-x_3, x-x_4) = \exp\Big\{ -i\Big[(x-x_1)\wedge(x-x_2) \\ &-(x-x_1)\wedge(x-x_3) + (x-x_1)\wedge(x-x_4) + (x-x_2)\wedge(x-x_3) \\ &-(x-x_2)\wedge(x-x_4) + (x-x_3)\wedge(x-x_4)\Big]\Big\}.\end{aligned} \tag{3.34}$$

Then the quantity

$$\begin{aligned}\Upsilon_h(t,\bar{x}) : &= \int d^D\bar{x}' \frac{\delta \mathcal{L}_h(t,\bar{x}')}{\delta \mathcal{Q}_h(t,\bar{x})} \cdot \frac{\epsilon(\bar{x}^0) - \epsilon(\bar{x}'^0)}{2} \\ &= -\delta(\bar{x}^0)\partial_{\bar{x}^0}\mathcal{Q}_h(t,\bar{x}) + \frac{\lambda}{4!}\int d^D y_1 d^D y_2 d^D y_3 d^D \bar{x}' \Big(\frac{\epsilon(\bar{x}^0) - \epsilon(\bar{x}'^0)}{2}\Big) \\ &\quad \times \mathcal{Q}_h(t,y_1)\mathcal{Q}_h(t,y_2)\mathcal{Q}_h(t,y_3)\Phi(x,y_1,y_2,y_3,y_4) \\ &+ \frac{\Omega^2}{8}\int d^D y_1 d^D y_2 d^D y_3 d^D \bar{x}'\, \tilde{y}_1\tilde{y}_2 \mathcal{Q}_h(t,y_3)\frac{\epsilon(\bar{x}^0) - \epsilon(\bar{x}'^0)}{2} \\ &\quad \times \Psi(x,y_1,y_2,y_3,y_4)\end{aligned} \tag{3.35}$$

with

$$\begin{aligned}\Phi(x,y_1,y_2,y_3,y_4) &= \Big[K(\bar{x}-\bar{x}', y_1-\bar{x}', y_2-\bar{x}', y_3-\bar{x}')\\ &+ K(y_1-\bar{x}', \bar{x}-\bar{x}', y_2-\bar{x}', y_3-\bar{x}')\\ &+ K(y_1-\bar{x}', y_2-\bar{x}', \bar{x}-\bar{x}', y_3-\bar{x}')\\ &+ K(y_1-\bar{x}', y_2-\bar{x}', y_3-\bar{x}', \bar{x}-\bar{x}')\Big] \end{aligned} \tag{3.36}$$

and

$$\begin{aligned}\Psi(x,y_1,y_2,y_3,y_4) &= \Big[K(\bar{x}'-y_1, \bar{x}'-\bar{x}, \bar{x}'-y_2, \bar{x}'-y_3)\\ &+ K(\bar{x}'-\bar{x}, \bar{x}'-y_1, \bar{x}'-y_3, \bar{x}'-y_2)\\ &+ K(\bar{x}'-y_1, \bar{x}'-y_3, \bar{x}'-\bar{x}, \bar{x}'-y_2)\\ &+ K(\bar{x}'-y_2, \bar{x}'-\bar{x}, \bar{x}'-y_1, \bar{x}'-y_3)\\ &+ K(\bar{x}'-y_1, \bar{x}'-y_3, \bar{x}'-y_2, \bar{x}'-\bar{x})\\ &+ K(\bar{x}'-y_3, \bar{x}'-y_1, \bar{x}'-\bar{x}, \bar{x}'-y_2)\\ &+ K(\bar{x}'-y_1, \bar{x}'-\bar{x}, \bar{x}'-y_3, \bar{x}'-y_2)\\ &+ K(\bar{x}'-y_2, \bar{x}'-y_3, \bar{x}'-y_1, \bar{x}'-\bar{x})\Big], \end{aligned} \tag{3.37}$$

allows to compute the family of primary constraints for the class of GW models defined by the parameter h as follows:

$$\Gamma_h(t,\bar{x}) \equiv \mathcal{P}_h(t,\bar{x}) - \Upsilon_h(t,\bar{x}) \approx 0. \tag{3.38}$$

The family of secondary constraints can be obtained in the same way. The previous lemma guarantees the convergence:

$$\mathrm{h} \to 0 \Rightarrow \mathcal{P}_h \to \mathcal{P}; \qquad \mathcal{Q}_h \to \mathcal{Q}; \qquad \Gamma_h \to \Gamma \tag{3.39}$$

as well as the limit of the family (3.38) of primary constraints:

$$\Gamma = \lim_{h\to 0}\Gamma_h = \mathcal{P}(t,\bar{x}) - \Upsilon(t,\bar{x}) \approx 0.$$

The secondary constraints appear as the equation of motion of the field $\mathcal{Q}_h$, i.e.

$$\Xi_h(t,\bar{x}) \approx 0. \tag{3.40}$$

The total Hamiltonian can be then defined as

$$\begin{aligned}\mathcal{H}^T_h(t,[\mathcal{Q}_h,\mathcal{P}_h]) = \mathcal{H}_h(t,[\mathcal{Q}_h,\mathcal{P}_h]) + \Lambda^1(t,\bar{x})\star\Gamma_h(t,\bar{x})\\ +\Lambda^2(t,\bar{x})\star\Xi_h(t,\bar{x}), \end{aligned} \tag{3.41}$$

where $\Lambda^i(t,\bar{x}), i=1,2$ are Lagrange multipliers. The corresponding field theory action $\mathcal{S}^T_h(t,\bar{x})$

$$\begin{aligned}\mathcal{S}^T_h(t,\bar{x}) &= \int dt d^D\bar{x}\,\Big(\mathcal{L}_h(t,\bar{x}) + \Lambda^1(t,\bar{x})\star\Gamma_h(t,\bar{x}) + \Lambda^2(t,\bar{x})\star\Xi_h(t,\bar{x})\Big)\\ &= \int dt d^D\bar{x}\,\mathcal{L}^T_h(t,\bar{x}), \qquad \Lambda^i(t,\bar{x})\in T^*J \end{aligned} \tag{3.42}$$

generates the Euler-Lagrange equation of motion

$$\begin{aligned}\frac{\delta\mathcal{S}_h^T(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')} &= \int dt d^D\bar{x}\Big(\frac{\delta\mathcal{L}_h(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')} + \Lambda^1(t,\bar{x})\star\frac{\delta\Gamma_h(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')}\\ &+ \frac{\delta\Lambda^1(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')}\star\Gamma_h(t,\bar{x}) + \Lambda^2(t,\bar{x})\star\frac{\delta\Xi_h(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')}\\ &+ \frac{\delta\Lambda^2(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')}\star\Xi_h(t,\bar{x})\Big) = 0\end{aligned} \tag{3.43}$$

which gives

$$\frac{\delta\mathcal{L}_h(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')} + \Lambda^1(t,\bar{x})\star\frac{\delta\Gamma_h(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')} + \Lambda^2(t,\bar{x})\star\frac{\delta\Xi_h(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')}\Big) \approx 0 \tag{3.44}$$

where the constraints equations (3.38) and (3.40) have been taken into account. If we perform the following set of infinitesimal transformations of simply connected continuous arbitrary group G:

$$\begin{aligned}&\bar{x}\longmapsto\bar{x}' = \bar{x} + \frac{1}{2}\Big(\varpi_a\star\frac{\delta\bar{x}}{\delta\varpi_a} + \frac{\delta\bar{x}}{\delta\varpi_a}\star\varpi_a\Big),\quad (a=1,2,\cdots)\\ &\mathcal{Q}_h(t,\bar{x})\longmapsto\mathcal{Q}_h^t(t,\bar{x}') = \mathcal{Q}_h(t,\bar{x}) + \frac{1}{2}\Big(\varpi_a\star\frac{\delta\mathcal{F}(\mathcal{Q}_h(t,\bar{x}))}{\delta\varpi_a} + \frac{\delta\mathcal{F}(\mathcal{Q}_h(t,\bar{x}))}{\delta\varpi_a}\star\varpi_a\Big),\end{aligned} \tag{3.45}$$

where $\mathcal{F}(\mathcal{Q}_h(t,\bar{x}))$ is a transformation of fields $\mathcal{Q}_h(t,\bar{x})$ and $\{\varpi_a(\bar{x})\}$ defines a family of infinitesimal parameters of this group, then the transformation $\mathcal{Q}_h^t(t,\bar{x}')$ of fields $\mathcal{Q}_h(t,\bar{x}')$ at a same point $\bar{x}'$ can be expressed through the generators G_μ^a as:

$$\mathcal{Q}_h^t(t,\bar{x}') = \Big(1 - \frac{i}{2}\{\varpi_a, G_\mu^a\}_\star + \mathbf{O}(\varpi^2)\Big)\star\mathcal{Q}_h(t,\bar{x}') = e_\star^{-\frac{i}{2}\{\varpi_a,G_\mu^a\}_\star}\star\mathcal{Q}_h(t,\bar{x}'), \tag{3.46}$$

with

$$e_\star^{i\alpha} = 1 + i\alpha + \frac{i^2}{2!}\alpha\star\alpha + \frac{i^3}{3!}\alpha\star\alpha\star\alpha + \ldots;\alpha\in C^\infty(\mathbb{R}) \tag{3.47}$$

and

$$\begin{aligned}\mathcal{Q}_h(t,\bar{x}') &= \mathcal{Q}_h\Big(t,\bar{x} + \frac{1}{2}(\varpi_a\star\frac{\delta\bar{x}}{\delta\varpi_a} + \frac{\delta\bar{x}}{\delta\varpi_a}\star\varpi_a)\Big)\\ &= \mathcal{Q}_h(t,\bar{x}) + \frac{1}{2}(\varpi_a\star\frac{\delta\bar{x}^\mu}{\delta\varpi_a} + \frac{\delta\bar{x}^\mu}{\delta\varpi_a}\star\varpi_a)\star\partial_\mu\mathcal{Q}_h(t,\bar{x}) + \mathbf{O}(\varpi^2).\end{aligned} \tag{3.48}$$

The group element $g = e_\star^{-\frac{i}{2}\{\varpi_a,G_\mu^a\}_\star}\in U_\star(N)$, where $U_\star(N)$ is the NC gauge group. The noncommutative generators G_μ^a are determined by the relation:

$$\begin{aligned}\frac{i}{2}\Big\{\varpi_a, G_\mu^a\Big\}_\star\star\mathcal{Q}_h(t,\bar{x}) &= \frac{1}{2}\Big\{\frac{\delta\bar{x}^\mu}{\delta\varpi_a},\varpi_a\Big\}_\star\star\partial_\mu\mathcal{Q}_h(t,\bar{x})\\ &\quad -\frac{1}{2}\Big\{\varpi_a,\frac{\delta\mathcal{F}(\mathcal{Q}_h(t,\bar{x}))}{\delta\varpi_a}\Big\}_\star.\end{aligned} \tag{3.49}$$

Let us now write the nonlocal Lagrangians (3.32) in the following form:

$$\begin{aligned}\mathcal{L}_h(t,\bar{x}) &= \mathcal{L}_h^\star(\mathcal{Q}_h(t,\bar{x}),\partial_\mu\mathcal{Q}_h(t,\bar{x}),\bar{x}) &(3.50)\\ &= \mathcal{L}_h\Big(\mathcal{Q}_h(t,\bar{x}),\partial_\mu\mathcal{Q}_h(t,\bar{x}),\partial_\mu\partial_\nu\mathcal{Q}_h(t,\bar{x}),\cdots,\bar{x};\Theta^{\alpha\beta}\Big). &(3.51)\end{aligned}$$

Remark that in equation (3.50), all products are the star ones and the EL equation of motion can be written in a similar form as in the usual commutative field theories:

$$\frac{\partial\mathcal{L}_h^\star}{\partial\mathcal{Q}_h}-\partial_\mu\frac{\partial\mathcal{L}_h^\star}{\partial\partial_\mu\mathcal{Q}_h}=0. \tag{3.52}$$

Setting $\zeta(\varpi,f)=\frac{1}{2}\Big(\varpi_a\star\frac{\delta f}{\delta\varpi_a}+\frac{\delta f}{\delta\varpi_a}\star\varpi_a\Big)$, then $\partial^t_\mu=\Big(\delta^\nu_\mu-\partial_\mu\zeta(\varpi,\bar{x})\Big)\partial_\nu$ and we can deduce the identity $d^D\bar{x}'=[1+\partial_\mu\zeta(\varpi,\bar{x})+\mathbf{O}(\varpi^2)]\mathrm{d}^D\bar{x}$. Using the relation (3.50), a direct evaluation of $\delta\mathcal{S}$ yields

$$\begin{aligned}\delta\mathcal{S} &= \mathcal{S}^t-\mathcal{S}\\ &= \int dtd^D\bar{x}'\,\mathcal{L}_h^\star(\mathcal{Q}_h^t(t,\bar{x}'),\partial^t_\mu\mathcal{Q}_h^t(t,\bar{x}'),\bar{x}')-\int dtd^D\bar{x}\,\mathcal{L}_h^\star(\mathcal{Q}_h(t,\bar{x}),\partial_\mu\mathcal{Q}_h(t,\bar{x}),\bar{x})\\ &= \int dtd^D\bar{x}\,\Big[(1+\partial_\mu\zeta(\varpi,\bar{x}))\star\mathcal{L}_h(t,\bar{x})+\zeta(\varpi,\mathcal{F})\star\frac{\partial\mathcal{L}_h(t,\bar{x})}{\partial\mathcal{Q}_h(t,\bar{x})}\\ &\quad+\zeta(\varpi,\bar{x})\star\partial_\mu(\mathcal{L}_h(t,\bar{x}))+\partial_\mu(\zeta(\varpi,\mathcal{F}))\star\frac{\partial\mathcal{L}_h(t,\bar{x})}{\partial\partial_\mu\mathcal{Q}_h(t,\bar{x})}\\ &\quad-\partial_\mu\zeta(\varpi,\bar{x})\partial_\nu\mathcal{Q}_h(t,\bar{x})\star\frac{\partial\mathcal{L}_h(t,\bar{x})}{\partial\partial_\mu\mathcal{Q}_h(t,\bar{x})}+\mathbf{O}(\varpi^2)\Big]-\int dtd^D\bar{x}\,\mathcal{L}_h(t,\bar{x})\\ &= \int dtd^D\bar{x}\,\Big[-\partial^\mu\mathcal{J}^a_\mu+\zeta(\varpi,\mathcal{F})\star\Big(\frac{\partial\mathcal{L}_h}{\partial\mathcal{Q}_h}-\partial_\mu\frac{\partial\mathcal{L}_h}{\partial\partial_\mu\mathcal{Q}_h}\Big)+\mathcal{B}(\varpi)\Big]. \end{aligned}\tag{3.53}$$

In this expression, the first term is a *divergence* term defining the NC tensor $\mathcal{J}^a_\mu$ expressed as follows:

$$\mathcal{J}^a_\mu=\frac{1}{4}\Big\{\Big\{\varpi_a,\frac{\delta\bar{x}^\nu}{\delta\varpi_a}\Big\}_\star,\mathcal{T}_{\mu\nu}\Big\}_\star-\frac{1}{4}\Big\{\Big\{\varpi_a,\frac{\delta\mathcal{F}(\mathcal{Q}_h(t,\bar{x}))}{\delta\varpi_a}\Big\}_\star,\frac{\partial\mathcal{L}_h(t,\bar{x})}{\partial\partial_\mu\mathcal{Q}_h(t,\bar{x})}\Big\}_\star. \tag{3.54}$$

The second term contains the EL equation of motion while the last term, usually called the breaking term, is given by the relation

$$\mathcal{B}(\varpi)=\frac{1}{4}\Big\{\zeta(\varpi,\bar{x}),\partial_\mu\Big(\{\partial_\nu\mathcal{Q}_h(t,\bar{x}),\frac{\partial\mathcal{L}_h(t,\bar{x})}{\partial\partial_\mu\mathcal{Q}_h(t,\bar{x})}\}_\star\Big)\Big\}_\star. \tag{3.55}$$

$\mathcal{T}_{\mu\nu}$ is the energy-momentum tensor computed in (3.30), defined with non-local variables $\mathcal{Q}_h(t,\bar{x})$:

$$\mathcal{T}_{\rho\mu}=\frac{1}{2}\{\partial_\rho\mathcal{Q}_h(t,\bar{x}),\partial_\mu\mathcal{Q}_h(t,\bar{x})\}_\star-g_{\rho\mu}\mathcal{L}_h^\star. \tag{3.56}$$

The translational invariance violation, engendered by the appearance of the coordinate $\tilde{\bar{x}}^\mu$, can be avoided by imposing the constraint

$$\pi(\mathcal{Q}(t,\bar{x}),\tilde{x})\equiv\delta\mathcal{S}(t,\bar{x})/\delta\tilde{\bar{x}}^\mu\approx 0. \tag{3.57}$$

It is worth noticing that if $\varpi_a \star (\delta x^\mu / \delta \varpi_a)$ is a constant parameter and $\mathcal{F}$ is trivial, then the current (3.54) is reduced to the NC energy momentum tensor (3.56). If $\varpi_a \star (\delta x^\mu / \delta \varpi_a)$ is defined as $\varpi_a^{\mu\nu} x_\nu$, where $\varpi_a^{\mu\nu}$ is the Lorentz tensor and $\varpi_a \star (\delta \mathcal{F} / \delta \varpi_a) = -\varpi_a^{\mu\nu} x_\nu \partial_\mu \mathcal{Q}_h(t, \bar{x})$, then the current (3.54) is reduced to the angular momentum tensor. The current $\mathcal{J}_\mu^a$ is not symmetric, nonlocally conserved, and in massless theory, not traceless.

4 Concluding remarks

We have provided a generalization of the Hamiltonian formulation developed by Gomis *et al* [3], which has been applied to the renormalizable Grosse-Wulkenhaar $\phi^4_{\star D}$ model. The Euler-Lagrange equation of motion has been derived. The constraints and NC currents have been investigated and analyzed. The following statements are worthy of attention:

1. It is possible to study the original D dimensional non-local Lagrangian system (3.1) describing the renormalizable GW model as a $D+1$ dimensional local (in one of the times) Hamiltonian system, governed by the Hamiltonian (3.41) and a set of constraints.

2. Examples of Hamiltonian symmetry generators of class of the renormalizable GW model working in a $D+1$ dimensional space can be given.

3. As expected from previous investigations on NC Noether currents, the tensor $\mathcal{J}_\mu^a$ (3.54) is not symmetric, nonlocally conserved, and, in massless theory, not traceless.

4. A characteristic feature of the Hamiltonian formalism for non-local theories is that it contains the Euler-Lagrange equations as Hamiltonian constraints. The Euler-Lagrange equation of motion is a constraint in the space of trajectories.

The EL equation of motion in $D+1$ dimensions can be also solved using the matrix base formalism. In that case, the matrix elements can be written as:

$$\mathcal{B}_{h,kl}^{(D+1)}(t, \bar{x}) = \int dt'\, \omega_h(t - t') e^{t \frac{d}{d\bar{x}}} \left(b_{kl}^{(D)}(\bar{x}) \right). \tag{4.1}$$

where $e^{t \frac{d}{d\bar{x}}}$ can be taken as the evolution operator T_t (translation operator). The fields $\mathcal{Q}_h(t, \bar{x})$ can be reexpressed as follows:

$$\mathcal{Q}_h(t, \bar{x}) = \sum_{k,l} C_{kl} \mathcal{B}_{h,kl}^{(D+1)}(t, \bar{x}). \tag{4.2}$$

Then, the formalism developed in [19] can be applied step by step. Further, the same matrix base method can be adapted to formulate the NC tensors $\mathcal{J}_\mu^a$. Unfortunately,

such a computation is too tedious and gives rise to cumbersome expressions that are irrelevant for this work. Moreover, their interpretation needs more investigations whose results will be in the core of a forthcoming paper.

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